The Mean and Variance Of The Sum Of A Normal And Exponentially-Distributed Random Variate

Gary Schurman MBE, CFA

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We are tasked with determining the mean and variance of the sum of a normally-distributed random variate and an exponentially-distributed random variate. We will assume that the correlation between the two random variates is zero (i.e. the random variates are independent). This white paper will come in handy when calculating the return mean and variance of a jump-diffusion process where we have a standard Brownian motion combined with a one jump event (example: loss of a major customer) that may or may not occur.

Our Hypothetical Problem

We pull a random variate from a normal distribution with mean 4.00 and variance 9.00, and pull a random variate from an exponential distribution with an expected arrival time of 8.00 years. With is the mean and variance of the sum of the random variates given that the random variates are independent?

Equations for Mean and Variance

Let's assume that we have two random variates x and y drawn from two different probability distributions. The mean and variance of the two random variates are...

$$x \sim D_x\left[m, v\right] \text{ ...and... } y \sim D_y\left[n, w\right] \tag{1}$$

Using Equation (1) above we will define the function $f(x, y)$ to be...

$$f(x, y) = x + y \tag{2}$$

Using Equations (1) and (2) above the equation for the first moment of the distribution of $f(x, y)$ is...

$$\mathbb{E}\left[f(x, y)\right] = \mathbb{E}\left[x + y\right] = \mathbb{E}\left[x\right] + \mathbb{E}\left[y\right] \tag{3}$$

Note that the square of Equation (3) above is...

$$\mathbb{E}\left[f(x, y)\right]^2 = \mathbb{E}\left[x\right]^2 + \mathbb{E}\left[y\right]^2 + 2\,\mathbb{E}\left[x\right]\mathbb{E}\left[y\right] \tag{4}$$

Using Equations (1) and (2) above the equation for the second moment of the distribution of $f(x, y)$ is...

$$\mathbb{E}\left[f(x, y)^2\right] = \mathbb{E}\left[(x + y)^2\right] = \mathbb{E}\left[x^2 + y^2 + 2xy\right] = \mathbb{E}\left[x^2\right] + \mathbb{E}\left[y^2\right] + 2\,\mathbb{E}\left[x\,y\right] \tag{5}$$

Using Equations (1) and (3) above the equation for the mean of the distribution of $f(x, y)$ is...

$$\text{Mean of } f(x, y) = \mathbb{E}\left[f(x, y)\right] = \mathbb{E}\left[x\right] + \mathbb{E}\left[y\right] = m + v \tag{6}$$

Note that the equations for the variance of x and y are...

$$\text{Variance of x} = \mathbb{E}\left[x^2\right] - \left[\mathbb{E}\left[x\right]\right]^2 \text{ ...and... } \text{Variance of y} = \mathbb{E}\left[y^2\right] - \left[\mathbb{E}\left[y\right]\right]^2 \tag{7}$$

Note that the equations for the covariance of x and y are...

$$\text{Covariance of x and y} = \mathbb{E}\Big[x\,y\Big] - \mathbb{E}\Big[x\Big]\,\mathbb{E}\Big[y\Big] \tag{8}$$

Using Equations (4) and (5) above the equation for the variance of the distribution of $f(x,y)$ is...

$$\text{Variance of } f(x,y) = \mathbb{E}\Big[f(x,y)^2\Big] - \mathbb{E}\Big[f(x,y)\Big]^2 = \mathbb{E}\Big[x^2\Big] - \mathbb{E}\Big[x\Big]^2 + \mathbb{E}\Big[y^2\Big] - \mathbb{E}\Big[y\Big]^2 + 2\bigg(\mathbb{E}\Big[x\,y\Big] - \mathbb{E}\Big[x\Big]\mathbb{E}\Big[y\Big]\bigg) \tag{9}$$

Using Equations (7) and (8) above we can rewrite Equation (9) above as...

$$\text{Variance of } f(x,y) = \text{Variance of x} + \text{Variance of y} + 2 \times \text{Covariance of x and y} \tag{10}$$

Since the random variates x and y are independent the covariance of x and y is zero. Using Equation (1) above we can rewrite Equation (10) above as...

$$\text{Variance of } f(x,y) = v + w + 2 \times 0 = v + w \tag{11}$$

The Answer To Our Hypothetical Problem

The first thing that we have to determin is the mean and variance of the exponentially-distributed random variate. Note that the mean and variance are...

$$\lambda = \frac{1}{8\text{ years}} = 0.1250 \;\text{ ...and... }\; \text{Mean} = \frac{1}{\lambda} = 8.00 \;\text{ ...and... }\; \text{Variance} = \frac{1}{\lambda^2} = 64.00 \tag{12}$$

Using Equations (6) and (12) above and the parameters to our hypothetical problem the mean of the distribution of the sum of the random variates is...

$$\text{Mean} = 4.00 + 8.00 = 12.00 \tag{13}$$

Using Equations (11) and (12) above and the parameters to our hypothetical problem the variance of the distribution of the sum of the random variates is...

$$\text{Mean} = 9.00 + 64.00 = 73.00 \tag{14}$$